COMPUTABLE NUMBERINGS ON THE APPROACH BY SORBI AND GONCHAROV

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We will discuss the results and problems from the approach of computable numberings.

Let S^* be a set of objects, and let L be some formal language. We can take some interpretation Int from L onto S^*.

A numbering ν from $\mathbb{N}$ onto S, where $S \subseteq S^*$, is *computable* (*R-computable*) if there is a computable (R-computable) function from $\mathbb{N}$ to L such that $\nu(n) = int(f(n))$ for any n.

A numbering ν is *reducible* to a numbering μ (denoted by $\nu \leq \mu$) if there is a computable function f such that $\nu(n) = \mu(f(n))$ for every n from $\mathbb{N}$. A numbering ν is *equivalent* to a numbering μ if $\nu \leq \mu$ and $\mu \leq \nu$.

Now we can define the Rogers semilattice $R(S, int)$: this upper semilattice is induced by the set of all R-computable numberings of a family S, and by the equivalence between numberings.

We discuss the main problems connected with the algebraic and model-theoretic properties of these semilattices, and connections of these semilattices for different classes.

Classical numberings.

Open problem. Is there a family of c.e. sets, which has exactly two (exactly $n+2$) minimal computable numberings? (Yu. Ershov)

Open problem. Let S and S' be finite families. In what cases the semilattices $R(S, \Sigma^0_1)$ and $R(S', \Sigma^0_1)$ are isomorphic?

Arithmetical and hyperarithmetical hierarchies. (Σ^0_α)

Open problem. Is there a family S such that $R(S, \Sigma^0_{n+2})$ is not isomorphic to $R(S_0, \Sigma^0_{n+1})$ for any Σ^0_{n+1}-computable family S_0?

Ershov hierarchy.

We will present some new results about Rogers semilattices in the Ershov hierarchy with minimal and maximal elements.

Theorem 1. For any $n \geq 2$, there is a family S of Σ^{-1}_n-sets such that the semilattice $R(S, \Sigma^{-1}_n)$ has two minimal elements which are Friedberg numberings, and $R(S, \Sigma^{-1}_n)$ has a maximal element.

Theorem 2. For any $n \geq 2$, there is a family S of Σ_n^{-1}-sets such that $R(S, \Sigma_n^{-1})$ has the least element which is a Friedberg numbering, and $R(S, \Sigma_n^{-1})$ has a maximal element.

Open problem. Is there a family S such that $R(S, \Sigma_{n+2}^{-1})$ has exactly $k \geq 3$ minimal elements?

Open problem. Is there a family S such that $R(S, \Sigma_{n+2}^{-1})$ has exactly $k \geq 2$ minimal (Friedberg) elements, and $R(S, \Sigma_{n+2}^{-1})$ does not have a maximal element?

We also consider families of computable functionals of finite types by Ershov. Yu. Ershov proved that in this case, there is a universal numbering. In addition, S. Ospichev costructed computable Friedberg numberings. What is the connection between classes of different types?

Open problem. Are there types $(\rho \to \tau)$ and $(\rho^* \to \tau^*)$, which are different, but the corresponding Rogers semilattices of all partial computable functionals are isomorphic? In which case they are not isomorphic?

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