Computable numberings on the approach by Sorbi and Goncharov.

S.S.Goncharov

Institute of Mathematics of SB RAS, Novosibirsk State University, Novosibirsk, Russia.

A satelitte event of the 8th European Congress of Mathematics16-18 June 2021, Udine, Italy..

E-mail: s.s.goncharov@math.nsc.ru

Partially supported by Grant RFBR of 20-01-00300

A comprehensive and extensive study of generalized computable numberings was initiated at the end of the past century, through a unifying approach towards a notion of a computable numbering for a family of sets of constructive objects, suggested in the paper of S.Goncharov and A.Sorbi.

A first program of such a study was outlined in the paper by S.A. Badaev, S.S. Goncharov. (*Theory of numberings: open problems.* In: Computability Theory and its Applications, P. Cholak, S. Lempp, M. Lerman and R. Shore eds.—Contemporary Mathematics, American Mathematical Society, 2000, vol. 257, pp. 23-38, Providence.) The first approach was connected with arithmetical hierarchy.

The series of papers was presented by S.Badaev, S.Goncharov, A.Sorbi, S.Podzorov. In these papers were studied the problem of structural properties of Rogers semilattices of families in classes of hierarchies of arithmetical and hyperarithmetical sets.

Generalization of computable numberings (Goncharov-Sorbi)

In the paper by S.S. Goncharov and A. Sorbi (*Generalized computable numberings and non-trivial Rogers semilattices.* Algebra and Logic, 1997, vol. 36, no. 6, pp. 359–369) was suggested some generalzation of the notion of computable numberings.

let S^* be some family of objects with description in a formal language L. We fix some interpretation Int of the language L in S^*.

A numberings ν from the set of natural numbers N in the set $S \subseteq S*$ we call **computable**, if there exists a computable function f from N in L such that $\nu(n) = int(f(n))$ for any $n \in N$.

We can consider some restriction on the classes of computable function from class F of function (polynomial, computable relative to oracle, or from some classes of hierarchies). We can fix this class F.

Generalization of computable numberings

We can define for subclass $S \subseteq S^*$ the set $\mathbf{R}(S, int, F)$ of all computable numberings relative to fix interpretation int for formal language L and class F of $computable$ functions. In the cases we have definable interpretation we will write without F, as $\mathbf{R}(S)$.

Generalization of computable numberings

If we have two F-computable numberings ν, μ from $\mathbf{R}(S, int, F)$ on S, we will say that ν F- **reducible to** μ ($\nu \leq_F \mu$, if there exists a function t from F such that $\nu(n) = \mu(t(n)$ for any $n \in N$.

In this case we have on **R** preorder $\leq_F$. Relative to this reducibility we have equivalence ($\nu =_F \mu$ if ($\nu \leq_F \mu$ and ($\mu \leq_F \nu$.

Let $R(S, int, F) \rightleftharpoons \mathbf{R}(S, int, F)/ =_F$ with partial order $\leq_F$.

The numbering $\nu \oplus \mu$ where $\nu \oplus \mu(2n) = \nu(n)$ and $\nu \oplus \mu(2n+1) = \mu(n)$ for any $n \in N$ is supremum of elements $\nu/ =_F$ and $\mu/ =_F$ in $(R(S, int, F), \leq_F)$.

This upper semilattice $(R(S, int, F), \leq_F)$ of F-computable numberings relative to interpretation int for L is called the Rogers semilattice.

Classical computable numberings

In classical numbering theory Yu. Ershov presented open problem about type of isomorphisms of Rogers semilattices on finite families of c.e. sets. S.Denisov constructed isomorphisms for finite families with smallest element and n another elements without inclusions.

Theorem (Ershov Yu. L.) If $R(S_0) \cong R(S_1)$ then $S_0' \cong S_1'$ where S' is the set of all essential elements of S with an order induced by the order $\subseteq$.

Ershov proved that some special classes of finite sets S_n with smallest element and all another without inclusions will be isomorphic to m-degrees.

Open problems

Open problem. Let S, S' be finite families. In what cases $R(S, \Sigma^0_1)$ and $R(S', \Sigma^0_1)$ are isomorphic? (Yu.L. Ershov)

Kudinov Hypothesis.

Open problem. Is there a family c.e. sets with exactly 2 (n+2) minimal computable numberings? (Yu.Ershov)

Open problem. What is about minimal non-positive elements in the Rogers semilattices with non-least Friedberg numbering?

Computable numberings in the levels of arithmetical hierarchy

Theorem

There is lattices of $\Sigma^0_m(S_0)$-computable numbering of families of Σ^0_m-sets which is non-isomorphic any $R^0_n(S)$ of for $m > n+1$.

Theorem

The lattice of Σ^0_m-computable numbering of families of all Σ^0_m-sets is non-isomorphic any $R^0_n(S)$ of for $m > n+1$.

The question for Rogers semilattices R^0_n of arithmetical Σ^0_n-computable sets and R^0_m of Σ^0_m-computable numberings was solved by Badaev-Goncharov-Sorbi $(m > n+2)$) and Podzorov $(m > n+1)$.

Open problem. Is there a family S such that $R(S, \Sigma^0_{n+3})$ is not isomorphic to $R(S_0, \Sigma^0_{n+2})$ for any Σ_{n+1}-computable family S_0.

Open problem. What about elementary theories of all Rogers semilattices for different levels and the Rogers semilattices of different levels?

Limit levels in the Knight hierarchy

Theorem

(D. Velegzhanina, P. Sagina)

1. There are infinitely many Fridberg arithmetical computable numberings of all arithmetical sets.

2. There are infinitely many Fridberg computable numberings of all sets from $\bigcup_{\beta <_O \alpha} \Sigma^0_\beta$ for computable limit ordinal $\alpha \in O^{CK}$.

Theorem

(D. Velegzhanina, P. Sagina)

The Rogers semilattice of all arithmetical computable numberings of all arithmetical sets is not isomorphic to any Rogers semilattice of any Σ^0_α-computable family of $\bigcup_{\beta <_O \alpha} \Sigma^0_\beta$ for computable limit ordinal α from O^{CK}.

Computable functionals(Yu. Ershov)

Topological spaces.

Let $\tau = (T, \sigma)$ be a topological T_0 space. We can define a partial order $x \leq y$ if for every open set V iff $x \in V$ than $y \in V$.

The open set V is f-set, if there is an element a_V such that $V = \{b \in T | a_V \leq b\}$, and we call a_V f-element (finite element).

We call the topological space f-space, if

(1) for any f-sets V_0 and V_1 the intersection $V_0 \cap V_1$ is f-element and

(2) The family of all f-sets with empty set is the basis of topology σ.

Definition. The topological space f-space is Ershov space (f_0-space) if the all set T is f-set and with finite elements T^0.

Continuous functions

Yu.L. Ershov, The theory of numberings and Yu. L. Ershov, The topology for discrete mathematics, 2020.

Let $\tau_0 = (T_0, \sigma_0)$ be Ershov space (f_0-space) and $\tau_1 = (T_1, \sigma_)$ be f-space..

We consider the set of continues functions $C(\tau_0, \tau_1)$ from τ_0 to τ_1. We will have f_0-space with basis topology of set form $U = \cap_{i<k}\langle V_i, W_i\rangle$ where $\langle V_i, W_i\rangle = \{\varphi \in C(\tau_0, \tau_1) | \varphi(V_i) \subseteq W_i\}$ where V_i and W_i are a f-sets.

Theorem. The set of function $C(\tau_0, \tau_1)$ with this topology is Ershov space with partial ordering: $\varphi_0 \leq \varphi_1$ iff $(\forall x).\varphi_0(x) \leq \varphi_1(x)$

with finite elements $C(\tau_0^0, \tau_1^0) \subseteq C(\tau_0, \tau_1)$

Theorem(Ershov Yu. L.) This class is closed relative direct products.

Posets and topologies

There is a dense connection of f-spaces with posets with subsets of finite elements.

Let T be a f-space and T_0 is a basic subspace of finite elements.

Consider the poset $(T, T_0, \leq)$ with partial ordering $\leq$ from this topology with subset of finite elements.

Properties of this structure:

1. $T_0 \subseteq T$;

2. $\leq$ is partial order on T;

3. for every elements x, y from T_0 if $(\exists z \in T)(x \leq z \& y \leq z)$ then there is $sup x, y$ in T_0;

4. $(\forall x)(x \in T \Rightarrow (\exists x_0 \in T_0)(x_0 \leq x)$;

5. $(\forall xy)((x, y \in T \& x \not\leq y) \Rightarrow (\exists x_0 \in T_0)(x_0 \leq x \& x_0 \not\leq y)$;

The structure $(T, T_0, \leq)$ with these properties can define the topological f-space with this structure and any element of T is a limit of finite elements.

Complete f-spaces.

Types of functionals by induction

.

1. 0 is a type.

2. If λ, μ are tupes, then $(\lambda \times \mu)$ and $(\lambda \to \mu)$ are types.

We have equivalences defined by induction:

1. $(\lambda \times \mu) \sim (\mu \times \lambda)$ and $((\lambda \times \mu) \times \rho) \sim (\mu \times (\lambda \times \varrho))$;

2.$((\lambda \times \mu) \to \rho) \sim (\mu \to (\lambda \to \varrho))$

3. $((\lambda \to (\mu \times \rho) \sim (\mu \to \lambda) \times (\mu \to \varrho))$

4. $(\mu \sim \mu')$ and $\lambda \sim \lambda' \Rightarrow (\mu \times \lambda) \sim (\mu' \times \lambda')$ and $(\mu \to \lambda) \sim (\mu' \to \lambda')$.

λ-model of Functionals

The class of functionals F_λ such that λ is type from T is λ-model if

there is the family of functionals $S_{(\mu,\mu')}|mu \sim \mu'$ such that

$S_{(\mu,\mu')} \in F_{(\mu,\mu')}$ and for any f from F_μ we have $S_{(\mu,\mu)} \circ f = f$ and

$S_{(\mu,\mu'')} \circ f = S_{(\mu',\mu'')} \circ (S_{(\mu,\mu')} \circ f))$ for $\mu \sim \mu' \sim \mu''$.

Yu. L. Ershov consider the P-problem about the existence of computable universal computable numberings for the set of morphisms $Mor((S_0, \nu_0), (S_1, \nu_1))$ for numbered sets (S_0, ν_0) and (S_1, ν_1). He constructed the theory numbered set with approximations.

The function φ from S_0 in S_1 is morphism from (S_0, ν_0) in (S_1, ν_1) if there is a computable function f such that $\varphi(\nu_0(n) = \nu_1(f(n))$ for every $n \in N$.

Theorem (Yu.L. Ershov) If the pair (S_0, ν_0) and (S_1, ν_1) has $\sigma_{2,0}$- property then $Mor((S_0, \nu_0), (S_1, \nu_1))$ has universal computable numberings and has the same properties.

Let T be the set of types and for all $\sigma \in T$ we define family of all partial computable functionals of type σ:

1. C_0 is some family with the property $\sigma_{2,0}$ (from Ershov book). For example is the family of all partial computable functions, family of all c.e. sets, $\{\{n\}|n \in \omega\} \bigcup \{\emptyset\}$
2. $C_{\sigma \times \tau} = C_\sigma \times C_\tau$
3. $C_{\sigma|\tau} = \mathfrak{Mor}(C_\sigma, C_\tau)$

Theorem (Ershov Yu.L.) If the numbered set (S, ν) has approximation with computable properties (Ershov conditions $\sigma_{2,0}$) then there is a λ-model of computable functionals over this numbered set.

We can start from the numbered set $F_\pi(N)$ with additional element $\emptyset$ and we will construct the λ-model of partial computable functionals.

Theorem

(Yu. L. Ershov) There is a universal computable numbering of all τ-computable functionals sets.

Theorem

(S.Ospichev) There is a Friedberg computable numbering of all τ-computable functionals sets.

Theorem

(S.Ospichev) If there is the computable Friedberg numbering of all elements from C_0 *then for any* $\sigma \in T$ *there is a computable Friedberg numbering of all partial computable functionals of type* σ*.*

Corollary *If the type* σ *has a subword* $\tau_1|\tau_2$*, then family of all partial computable functionals of type* σ *has*

1. *infinitely many nonequivalent friedberg numberings*
2. *infinitely many nonequivalent positive undecidable numberings*
3. *infinitely many nonequivalent minimal numberings*

Open problem. Are there types $(\rho \to \tau)$ and $(\rho^* \to \tau^*)$, which are different, but the corresponding Rogers semilattices of all partial computable functionals are isomorphic? In which case they are not isomorphic?

Open problem. Is it possible to prove all properties of classical computable numberings for functionals?

Analitical hierarhy, James C. OWINGS, JR

Theorem

There is not a meta-c.e. numbering S(α)($\alpha < \omega_1$) Π^1_1-sets such that it is Friedberg numbering of all Π^1_1-sets.

Theorem

There is not a Σ^1_1-numbering Σ^1_1-sets such that it is a Friedberg numbering of all Σ^1_1-sets for $1 \leq n \leq 2$.

Analytical hierarchy, M. Dorzhieva

Theorem

There is not a Σ^1_n-numbering Σ^1_n-sets such that it is Friedberg numbering of all Σ^1_n-sets for $1 \leq n \leq 2$.

Proof is without metarecursion.

Theorem

(M. Dorzhieva, 2018) There is not a Σ^1_n-numbering Σ^1_n-sets such that it is Friedberg numbering of all Σ^1_n-sets for $n \geq 3$, if we have the axiom of constructibility.

There are same results by N. A. Bazhenov, M. Mustafa, S. S. Ospichev, M. M. Yamaleev, ¥Numberings in the analytical hierarchy¥, Algebra Logika, 59:5 (2020), 594Џ599 mathnet; Algebra and Logic, 59:5 (2020), 404Џ407 with another set-theoretical properties.

What about this question without additional axiom.

Theorem (Khutoretskii's Theorem)

Let S be a family of computably enumerable sets.

1. *If $\mu \not\leq \nu$ are computable numberings of S then there is a computable numbering π of S with $\pi \not\leq \nu$ and $\mu \not\leq \pi \oplus \nu$.*
2. *If the Rogers semilattice $R_1^0(S)$ of S contains more than one element, then it is infinite.*

Badaev-Lempp result in Computable numberings in Ershov hierarchy

Theorem (Badaev S.A. and Steffen Lempp, A decomposition of the Rogers semilattice of a family of d.c.e. sets. The Journal of Symbolic Logic, Vol. 74, No. 2, 2009)

There is a family $\mathcal{F}$ of d.c.e. sets, and there are computable numberings μ and ν of the family $\mathcal{F}$ such that for any computable numbering π of $\mathcal{F}$, either $\mu \leq \pi$ or $\pi \leq \nu$. In addition, we can ensure the following:

- *$\mathcal{F}$ is a family of c.e. sets and ν is a computable numbering of $\mathcal{F}$ as a family of c.e. sets;*
- *both μ and ν can be made Friedberg and thus minimal numberings; and so*
- *any computable numbering π of $\mathcal{F}$ satisfies $\pi \equiv \nu$ or $\mu \leq \pi$.*

Ershov hierarchy and principal and minimal computable numberings

There are series of research on these problems:

Theorem

(Talasbaeva) For every even $n \geq 2$ *(odd* $n \geq 1$*), an infinite* Σ_n^{-1}*-computable family* $\mathcal{S} \subseteq \Sigma_n^{-1}$ *admitting at least one* Σ_n^{-1}*-computable numbering and containing the empty set (the set* N*) possesses infinitely many non-equivalent* Σ_n^{-1}*-computable positive undecidable numberings.*

Theorem

(Badaev-Talasbaeva)
For every $n \in \omega \cup \{\omega\}$*,* $n \neq 0$*, there exist a family* $\mathcal{A}$ *consisting of n c.e. sets whose Rogers semilattice* $\mathcal{R}_2^{-1}(\mathcal{A})$ *is one-element.*

Theorem

(Badaev-Talasbaeva) There exists a family consisting of two embedded Σ_2^{-1}*-sets whose Rogers semilattice is one-element.*

S.Ospichev, Infinite family of Σ_a^{-1}-sets with a unique computable numbering, Journal of Mathematical Sciences, 2013, 188:4, 449Џ451

S. Ospichev, Computable family of Σ_a^{-1}-sets without Friedberg numberings, in: 6th Conference on Computability in Europe, CiE 2010, Ponta Delgada, Azores, Portugal, June/July 2010. Abstract and Handout Booklet, edited by F. Ferreira, H. Guerra, E. Mayordomo, and J. Rasga (University of Azores, 2010), pp. 311–315.

S.S.Ospichev, Some Properties of Numberings in Various Levels in Ershov's Hierarchy, Journal of Mathematical Sciences, 2013, 188:4, 441Џ448

S. S. Ospichev, Computable families of sets in Ershov hierarchy without principal numberings, Journal of Mathematical Sciences, 2016, 215:4, 529Џ536

Kalmurzaev B.S., Embeddability of the semilattice L_m^0 in Rogers semilattices, Algebra and Logic, Vol. 55, No. 3, 2016.

Ershov hierarchy and principal and minimal computable numberings

For infinite computable ordinals.

Theorem

(Abeshev-Badaev) For every ordinal notation ξ of a nonzero computable ordinal, there exists a Σ^{-1}_{ξ}-computable family whose Rogers semilattice has exactly one Friedberg degree which is not the least element of the semilattice.

Theorem

(Kalmurzaev) Let $\alpha + 1$ be an arbitrary nonlimit computable ordinal, b be any ordinal notation for α, and a be any ordinal notation for $\alpha + 1$. Suppose that A and B are Σ^{-1}_{a}-sets for which $B = B_0 \setminus \widetilde{B}$, where $B_0 \in \Sigma^{-1}_{1}$ and $\widetilde{B} \in \Sigma^{-1}_{b}$, and $B' = z : \exists t(g_B(z,t) = 1)$, where g_B is an ordinal function for B. Assume also that there exists a c.e. set C such that:

1. *$\widetilde{B} \cap A$ and $C \cap A$ are computable sets;*
2. *$C \supset B \setminus A$;*

Open questions in Ershov hierarchy

It is interesting questions about computable numberings relative to classes of Ershov Hierarchy.

Open problem. Are there for every $n \in N$ a families of Σ_2^{-1}-sets with exactly $n \geq 2$ Σ_2^{-1}-computable numberings.

Open problem. Is there a family with nontrivial finite Rogers semilattice in Ershov hierarchy.

Open problem. Is there a family of Σ_{n+3}^{-1}-sets which is not isomorphic to any Rogers semilattice Σ_{n+2}^{-1}-sets

Ershov hierarchy

We will present some new results about Rogers semilattices in the Ershov hierarchy with minimal and maximal elements together with S. Badaev and A. Sorbi.

Theorem 1. For any $n \geq 2$, there is a family S of Σ_n^{-1}-sets such that the semilattice $R(S, \Sigma_n^{-1})$ has two minimal elements which are Friedberg numberings, and $R(S, \Sigma_n^{-1})$ has a maximal element.

Theorem 2. For any $n \geq 2$, there is a family S of Σ_n^{-1}-sets such that $R(S, \Sigma_n^{-1})$ has the least element which is a Friedberg numbering, and $R(S, \Sigma_n^{-1})$ has a maximal element.

We used the paper "Reductions between Types of Numberings"by Ian Herbert, Mustafa Manat and Frank Stephan.

Open questions.

1. Is there families with $n > 2$ minimal elements in Rogers semilattices for finite levels of Ershov hierarchy.

2. Is it true that for finite sets of S the Rogers semilattice $R(S, \Sigma^{-1}_{n+2})$ is trivial or infinite?

The Ershov lattice.

Let U be the set $\{(X,Y)|X \subseteq N, Y \subseteq N, X \cap Y = \emptyset\}$

Let $(X,Y), (X',Y')$ be elements of U.
The element (X,Y) is sm-reducible to (X',Y') ($(X,Y) \leq_{sm} (X',Y')$) , iff there exist a computable function f such that $f(X) \subseteq X'$ and $f(Y) \subseteq Y'$.
And $(X,Y) \equiv_{sm} (X',Y')$, if $(X,Y) \leq_{sm} (X',Y')$ and $(X',Y') \leq_{sm} (X,Y)$.

The structure $(U/\equiv_{sm}, \leq_{sm})$ is a lattice.

Problem. Is it true that the theory of second ordered arithmetic is definable in the theory of the Ershov lattice $(U/\equiv_{sm}, \leq_{sm})$? (S.G. Dvornikov)