A Satellite Event of the 8th European Congress of Mathematics

On A-computable Families: Numberings, Rogers and Degtev Semilattices

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Equivalences, Numberings, Reducibilities

16–18 June 2021

Basic Definitions

Numbering of a countable set S is a surjective mapping $\nu : \mathbb{N} \longrightarrow S$.

Let $H(S)$ be the set of all numberings of S.

Definition

A numbering ν of a countable family $\mathcal{S} \subseteq 2^{\mathbb{N}}$ is computable (A-computable), if the set $G_\nu = \{\langle x, y\rangle : y \in \nu(x)\}$ is c.e. (A-c.e.). In this case, the family $\mathcal{S}$ is said to be also computable (A-computable).

If $A = \emptyset^{(n)}$, then A-computable numberings are called Σ^0_{n+1}-computable.

Let $\mathrm{Com}^A(\mathcal{S}) = \{\nu \in H(\mathcal{S}) : \nu \text{ is } A\text{-computable}\}$,
$\mathrm{Com}^0_{n+2}(\mathcal{S}) = \mathrm{Com}^{\emptyset^{(n+1)}}(\mathcal{S})$, $\mathrm{Com}(\mathcal{S}) = \mathrm{Com}^{\emptyset}(\mathcal{S})$.

History

- K. Gödel and A. Turing codings in logic
- A.N. Kolmogorov, foundations of Numbering Theory, mid 50s
- V.A. Uspensky, study of computable numberings of the family of all p.c. functions, 1955–1957
- H. Rogers, study of upper semilattices of equivalence classes of computable numberings of the family of all p.c. functions, 1958
- M.B. Pour-El, A.I. Maltsev, Yu.L. Ershov, S.A. Badaev, S.S. Goncharov. . . , study of numberings of families of c.e. sets, p.c. functions, computable models, positive equivalences, etc., since 1960

History

- S.S. Goncharov, A. Sorbi, approach to the notion of generalized computable numbering, 1997
- S.S. Goncharov, A. Sorbi, S.A. Badaev, S.Yu. Podzorov, S. Lempp. . . , study of computable numberings in well-known hierarchies, since 1997
- S.A. Badaev, S.S. Goncharov («Generalized computable universal numberings», Algebra and Logic, 53:5 (2014), 355–364), study of computable numberings with an oracle

Part I. Rogers Semilattices of *A*-computable Families

Basic Definitions

Let $\nu_0, \nu_1 \in H(S)$.

Definition

We say that ν_0 is reducible to ν_1 ($\nu_0 \leqslant \nu_1$) if $\nu_0 = \nu_1 \circ f$ for some computable function f. Numberings ν_0 and ν_1 are called equivalent ($\nu_0 \equiv \nu_1$) if $\nu_0 \leqslant \nu_1$ and $\nu_1 \leqslant \nu_0$.

Let $(\nu_0 \oplus \nu_1)(2x+i) = \nu_i(x)$, $i = 0, 1$.

Let $\mathcal{S}$ be an A-computable family.

The quotient structure $\mathscr{L}^A(\mathcal{S}) = \langle \mathrm{Com}^A(\mathcal{S})_{/\equiv}; \leqslant \rangle$ is said to be the Rogers semilattice of the family $\mathcal{S}$.

Let $\mathscr{L}^0_{n+2}(\mathcal{S}) = \mathscr{L}^{\emptyset^{(n+1)}}(\mathcal{S})$, $\mathscr{L}(\mathcal{S}) = \mathscr{L}^{\emptyset}(\mathcal{S})$.

Rogers Semilattices

A family $\mathcal{F}$ is effectively discrete if there is a computable function f s.t.

1. for every $X \in \mathcal{S}$ there exists an x with $D_{f(x)} \subseteq X$;
2. $D_{f(x)} \subseteq X \,\&\, D_{f(x)} \subseteq Y \Rightarrow X = Y$ for all $X, Y \in \mathcal{S}$ and $x \in \mathbb{N}$.

Properties of some Rogers semilattices

- **(Mal'tsev, 1961)** Let $\mathcal{S}$ be a computable effectively discrete family. Then $|\mathcal{L}(\mathcal{S})| = 1$.
- **(Goncharov, Sorbi, 1997)** Let $\mathcal{S}$ be a nontrivial ($|\mathcal{S}| > 1$) Σ^0_{n+2}-computable family. Then $\mathcal{L}^0_{n+2}(\mathcal{S})$ is infinite.
- **(Badaev, Goncharov, 2001)** Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family. Then $\mathcal{L}^0_{n+2}(\mathcal{S})$ has infinitely many minimal elements.

- **(Lachlan, 1964)** If $\mathcal{S}$ is a finite family of c.e. sets, then $\mathcal{L}(\mathcal{S})$ has the greatest element.
- **(Badaev, Goncharov, 2014)** Let $\emptyset' \leqslant_T A$. Then $\mathcal{L}^A(\mathcal{S})$, for a finite family of A-c.e. sets $\mathcal{S}$, has the greatest element iff $\bigcap \mathcal{S} \in \mathcal{S}$.

Let $\mathcal{L}$ be an upper semilattice.

The local isomorphism type of $\mathcal{L}$ = All isomporfism types of principle ideals of $\mathcal{L}$.

Rogers Semilattices

- **(Lachlan, 1972; Podzorov, 2008)** The class of all principal ideals of $\mathcal{L}(\mathcal{S})$, where $\mathcal{S}$ is a finite family of c.e. sets and $\mathcal{S}$ contains (does not contain) two different comparable under inclusion sets, coincides with the class of all bounded distributive upper semilattices with Σ^0_3-presentations (one-element semilattices).
- **(Podzorov, 2008)** The class of all principal ideals of $\mathcal{L}^0_{n+2}(\mathcal{S})$, where $\mathcal{S}$ is a finite nontrivial family of Σ^0_{n+2}-sets and $\mathcal{S}$ contains (does not contain) two different comparable under inclusion sets, coincides with the class of all bounded distributive upper semilattices with Σ^0_{n+4}-presentations (Σ^0_{n+3}-presentations).

Cardinality and Latticeness of $\mathscr{L}(\mathcal{S})$

Questions (Ershov, 1967)

1. What can we say about the cardinalities of the Rogers semilattices?
2. When are they lattices?

Theorem (Khutoretskii, 1971)

If $|\mathscr{L}(\mathcal{S})| > 1$, then $\mathscr{L}(\mathcal{S})$ is infinite.

Theorem (Selivanov, 1976)

If $|\mathscr{L}(\mathcal{S})| > 1$, then $\mathscr{L}(\mathcal{S})$ is not a lattice.

Cardinality and Latticeness of $\mathscr{L}^0_{n+2}(\mathcal{S})$

Theorem (Goncharov, Sorbi, 1997)

1. Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an infinite subset such that any two different elements of the subset form a minimal pair.
2. Let $\mathcal{S}$ be a finite family of Σ^0_{n+2}-sets such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an ideal that is isomorphic to the upper semilattice of c.e. m-degrees $\mathscr{L}^0$.

Corollary (Goncharov, Sorbi, 1997; Ershov, 1969)

Let $\mathcal{S}$ be a Σ^0_{n+2}-computable family such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ is infinite and not a lattice.

Cardinality and Latticeness of $\mathscr{L}^A(\mathcal{S})$

Let $\emptyset <_T A$.

Theorem

1. Let $\mathcal{S}$ be an infinite A-computable family. Then $\mathscr{L}^A(\mathcal{S})$ contains an infinite subset such that any two different elements of the subset form a minimal pair.
2. Let $\mathcal{S}$ be a finite family of A-c.e. sets such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^A(\mathcal{S})$ contains an ideal that is isomorphic to the following ideal of the upper semilattice of m-degrees: $I^m_T(A) = \{\deg_m(X) : X \leqslant_T A\}$.
3. The ideal $I^m_T(A)$ is not a lattice.

Theorem (Jockush, 1969)

The set $\{\deg_m(X) : X \equiv_{tt} A\}$ is infinite. Hence, $I^m_T(A)$ is also infinite.

Distinguishing $\mathscr{L}$ from $\mathscr{L}^A$

Let $\mathrm{R}_0 = \{\mathscr{L}(\mathcal{S}) : \mathcal{S} \text{ is computable}\}$, $\mathrm{Th}(\mathrm{R}_0) = \bigcap_{\mathfrak{A} \in \mathrm{R}_0} \mathrm{Th}(\mathfrak{A})$,

$\mathrm{R}_1 = \bigcup_{\emptyset <_T A} \{\mathscr{L}^A(\mathcal{S}) : \mathcal{S} \text{ is } A\text{-computable}\}$, $\mathrm{Th}(\mathrm{R}_1) = \bigcap_{\mathfrak{A} \in \mathrm{R}_1} \mathrm{Th}(\mathfrak{A})$.

Is there a difference between $\mathrm{Th}(\mathbf{R}_0)$ and $\mathrm{Th}(\mathbf{R}_1)$?

Definition

An upper semilattice $\langle L; \vee, \leqslant \rangle$ is (weakly) distributive if for every $a_0, a_1, b \in L$ with $b \leqslant a_0 \vee a_1$ (and $b \nleqslant a_0$, $b \nleqslant a_1$) there exist $b_0, b_1 \in L$ such that $b_0 \leqslant a_0$, $b_1 \leqslant a_1$ and $b = b_0 \vee b_1$.

Proposition (Folklore)

If $\mathcal{S}$ is a finite family of A-c.e. sets, then $\mathscr{L}^A(\mathcal{S})$ is distributive.

Distinguishing $\mathscr{L}$ from $\mathscr{L}^A$

Theorem (Badaev, Goncharov, Sorbi, 2003)

If $\mathcal{S}$ is an infinite Σ^0_{n+2}-computable family, then $\mathscr{L}^0_{n+2}(\mathcal{S})$ is not weakly distributive.

Theorem

1. If $\mathcal{S}$ is an infinite A-computable family, where $\emptyset <_T A$, then $\mathscr{L}^A(\mathcal{S})$ is not weakly distributive.
2. There is a computable family $\mathcal{S}$ such that $\mathscr{L}(\mathcal{S})$ is weakly distributive but not distributive.

Corollary

$\mathrm{Th}(\mathbf{R}_0) \neq \mathrm{Th}(\mathbf{R}_1)$.

Universal A-computable Numberings

A numbering $\nu \in \mathrm{Com}^A(\mathcal{S})$ is universal if $\alpha \leqslant \nu$ for each $\alpha \in \mathrm{Com}^A(\mathcal{S})$.

Theorem (Lachlan, 1964)

Any finite family of c.e. sets has a universal computable numbering.

Theorem (Badaev, Goncharov, 2014)

Let $\emptyset' \leqslant_T A$. Let $\mathcal{S}$ be a finite family of A-c.e. sets. Then $\mathcal{S}$ has a universal A-computable numbering iff $\bigcap \mathcal{S} \in \mathcal{S}$.

Question (Badaev, Goncharov, 2014)

What happens if we replace the set $\emptyset' \leqslant_T A$ by $\emptyset <_T A <_T \emptyset'$ or $A \mid_T \emptyset'$?

Universal A-computable Numberings

If $\emptyset <_T A \leqslant_T \emptyset'$ or $\emptyset' \leqslant_T A$, then $\deg_T(A)$ is hyperimmune.

Theorem

For a set A the following conditions are equivalent.

1. $\deg_T(A)$ is hyperimmune;
2. Let $\mathcal{S}$ be a finite family of A-c.e. sets. Then $\mathcal{S}$ has a universal A-computable numbering iff $\bigcap \mathcal{S} \in \mathcal{S}$.
3. There is a finite family of A-c.e. sets without universal A-computable numberings.

Corollary

Let $\deg_T(A)$ is hyperimmune-free. Then any finite family of A-c.e. sets has a universal A-computable numbering.

Khutoretskii Theorem

Theorem (Khutoretskii, 1971)

Let $a, c \in \mathscr{L}(\mathcal{S})$ and $c \not\leq a$. Then there exists $b \in \mathscr{L}(\mathcal{S})$ such that $a < b$ and $c \not\leq b$. In particular, if c is the greatest element of $\mathscr{L}(\mathcal{S})$ then c is limit: $\forall a \in \mathscr{L}(\mathcal{S}) \exists b \in \mathscr{L}(\mathcal{S})\, [a < c \Rightarrow a < b < c]$.

Theorem (Podzorov, 2004)

Let c be the greatest element of $\mathscr{L}^0_{n+2}(\mathcal{S})$. Then c is limit if one of the following conditions is met.

1. $\bigcap \mathcal{S} \in \mathcal{S}$;
2. $\mathcal{S}$ is finite;
3. $\mathcal{S}$ is Σ^0_{n+1}-computable.

Khutoretskii's Theorem

1. Is Khutoretskii Theorem true for $\mathscr{L}^A(\mathcal{S})$, or at least for $\mathscr{L}^0_{n+2}(\mathcal{F})$?
2. Is the greatest element of $\mathscr{L}^A(\mathcal{S})$ (if it exists), or at least of $\mathscr{L}^0_{n+2}(\mathcal{F})$ limit?

Let $\emptyset <_T B \leqslant_T A$.

A numbering $\alpha \in \mathrm{Com}^A(\mathcal{S})$ is said to be *B-universal* if for every $\beta \in \mathrm{Com}^A(\mathcal{S})$ there is a function $f \leqslant_T B$ such that $\beta = \alpha \circ f$.

Theorem

If $\alpha \in \mathrm{Com}^A(\mathcal{S})$ is not B-universal, then there exist $\beta_0, \beta_1 \in \mathrm{Com}^A(\mathcal{S})$ such that $\alpha < \beta_0$, $\alpha < \beta_1$ and $[\alpha] = [\beta_0] \wedge [\beta_1]$.

Corollary

Let $a, c \in \mathscr{L}^A(\mathcal{S})$ and $c \nleqslant a$, where $a = [\alpha]$ for some non-B-universal $\alpha \in \mathrm{Com}^A(\mathcal{S})$. Then there exists $b \in \mathscr{L}^A(\mathcal{S})$ such that $a < b$ and $c \nleqslant b$.

Khutoretskii Theorem

A numbering γ is non-splittable if there are no numberings $\gamma_0, \gamma_1 < \gamma$ of subsets of $\gamma(\mathbb{N})$ such that $\gamma \equiv \gamma_0 \oplus \gamma_1$.

It is known (see Selivanov's (1982) and Badaev and Goncharov's (2014) papers) that if $\emptyset' \leqslant_T A$ and $\gamma \in \mathrm{Com}^A(\mathcal{S})$ is universal, then γ is non-splittable.

Let $\gamma \in \mathrm{Com}^A(\mathcal{S})$ be a non-splittable numbering, $[\gamma] = c \in \mathscr{L}^A(\mathcal{S})$, $a \in \mathscr{L}^A(\mathcal{S})$.

Theorem

If A is high over B ($B \leqslant_T A$, $B'' \leqslant_T A'$), γ is B-universal and $c \nleqslant a$, then there exists $b \in \mathscr{L}^A(\mathcal{S})$ such that $a < b$ and $b \nleqslant c$.

Corollary

If $\emptyset' \leqslant_T A$ and $a < c$, then there exists $b \in \mathscr{L}^A(\mathcal{S})$ such that $a < b$ and $b \nleqslant c$. In particular, the greatest element of $\mathscr{L}^A(\mathcal{S})$ and, therefore, $\mathscr{L}^0_{n+2}(\mathcal{F})$ (if it exists) is limit.

Minimal Numberings

A numbering $\mu \in H(S)$ is minimal if $\alpha \leqslant \mu \Rightarrow \mu \leqslant \alpha$ for each $\alpha \in H(S)$.

Let $\eta_\mu = \{\langle x, y\rangle : \mu(x) = \mu(y)\}$. We say that μ is positive if η_μ is c.e. and μ is single-valued if η_μ is "=".

Let $K_{sv}(S)$, $K_{pos}(S)$ and $K_{min}(S)$ be the classes of all single-valued, positive, and minimal numberings of S respectively; $K_{sv}(S) \subseteq K_{pos}(S) \subseteq K_{min}(S)$.

Definition (S. Goncharov, A. Yakhnis, V. Yakhnis, 1993)

A class $\mathscr{C} \subseteq H(\mathcal{S})$ is A-effectively infinite if there is a p.c. function ψ such that
$\{\alpha^A_{\varphi_e(x)} : \varphi_e(x)\downarrow\} \subseteq \mathscr{C} \Rightarrow \forall x \in \mathrm{dom}\varphi_e\, [\mathscr{C} \ni \alpha^A_{\psi(e)} \not\equiv \alpha^A_{\varphi_e(x)}]$,
for each e, where α^A_n is the A-computable numbering with the Gödel number n.

Minimal Numberings

Let $\mathscr{E}^A = \{X \subseteq \mathbb{N} : X \text{ is } A\text{-c.e.}\}$ and $\mathscr{E} = \mathscr{E}^\emptyset$.

Theorem (S. Goncharov, A. Yakhnis, V. Yakhnis, 1993)

The classes $K_{sv}(\mathscr{E})$, $K_{pos}(\mathscr{E})$, $K_{min}(\mathscr{E})$ are effectively infinite.

Corollary

$K_{sv}(\mathscr{E}^A)$ is A-effectively infinite for each A, and $K_{pos}(\mathscr{E}^{\emptyset^{(n)}})$ is $\emptyset^{(n)}$-effectively infinite for each $n \in \mathbb{N}$.

Theorem (Badaev, Goncharov, 2001)

Any infinite Σ^0_{n+2}-computable family has infinitely many Σ^0_{n+2}-computable minimal numberings.

Minimal Numberings

Question (Badaev, Goncharov, 2001)

Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family. Is the class $K_{min}(\mathcal{S}) = \{\mu \in H(\mathcal{S}) : \mu \text{ is minimal}\}$ $\emptyset^{(n+1)}$-effectively infinite?

Theorem

Let A be a high set ($\emptyset'' \leqslant_T A'$). Then for any infinite A-computable family $\mathcal{S}$ we have
$\overline{A'''} \leqslant_1 \mathrm{Min}^A(\mathcal{S}) = \{e : \alpha^A_e \in K_{min}(\mathcal{S})\}$.

Corollary

Let A be a high set (in particular, $A = \emptyset^{(n+1)}$) and $\mathcal{S}$ an infinite A-computable family. Then $K_{min}(\mathcal{S})$ is A-effectively infinite.

Part II. Upper Semilattices of X-computable Families

The Upper Semilattice Ω^X

Let $\Omega^X = \{\mathcal{S} \subseteq 2^{\mathbb{N}} : \mathcal{S}$ is X-computable$\}$. Then $\langle \Omega^X; \subseteq \rangle$ is an upper semilattice with the greatest element $\mathcal{E}^X$ and the least element $\emptyset$.

Degtev, A.N. The semilattice of computable families of recursively enumerable sets. Mathematical Notes of the Academy of Sciences of the USSR 50, 1027–1030 (1991).

The families $\mathcal{S}_0 = \{\{2x\} : x \notin X'\} \cup \{\mathbb{N}\}$, $\mathcal{S}_1 = \{\{2x\} : x \notin X'\} \cup \{2\mathbb{N}\}$ have no infimum in Ω^X. Therefore, Ω^X is not a lattice.

Any element $\mathcal{S} \in \Omega^X$ is an atom of Ω^X iff $|\mathcal{S}| = 1$. Therefore, the Fréchet ideal I_F^X of Ω^X is equal to the class $\mathrm{Fin}^X = \{\mathcal{F} \in \Omega^X : \mathcal{F}$ is finite$\}$.

Minuends

Definition

A family $\mathscr{A} \in \Omega^X$ is a minuend if $\mathscr{A} \setminus \mathscr{B} \in \Omega^X$ for any $\mathscr{B} \in \Omega^X$.
Let Ω^X_M be the class of all minuends.

Definability of Ω^X_M: $\mathscr{A} \in \Omega^X_M$ iff $\forall \mathscr{B} \in \Omega^X \exists \mathscr{C} \in \Omega^X [\mathscr{C} \subseteq \mathscr{A} \,\&\, \mathscr{A} \cup \mathscr{B} = \mathscr{C} \cup \mathscr{B} \,\&\, \forall \mathscr{D} \in \Omega^X [\mathscr{D} \subseteq \mathscr{C} \,\&\, \mathscr{D} \subseteq \mathscr{B} \Rightarrow \mathscr{D} = \emptyset]]$.

Theorem (Degtev, 1991)

Ω^X_M is an ideal of Ω^X that forms a lattice.

Theorem

$\Omega^X_M = \mathrm{Fin}^X = \{\mathscr{F} \in \Omega^X : \mathscr{F} \text{ is finite}\}$. Therefore, I^X_F is definable in Ω^X.

Minuends

A numbering ν is precomplete if for every p.c. function ψ there exists a computable function f such that $\nu(\psi(n)) = \nu(f(n))$ if $\psi(n)\downarrow$.

Theorem (Ershov, 1977)

A numbering ν is precomplete iff for any p.c. function ψ there is a computable function g s.t. $\nu(g(x)) = \nu(\psi(x, g(x)))$, if $\psi(x, g(x))\downarrow$.

Theorem

Let $\mathcal{S} \in \Omega$ and $|\mathcal{S}| > 1$. If $\mathcal{S}$ has a precomplete, positive, universal computable numbering, then there is an infinite family $\mathcal{A} \subseteq \mathcal{S}$ such that

1. $\mathcal{A} \setminus \mathcal{B}$ is finite for each infinite computable family $\mathcal{B} \subseteq \mathcal{A}$;
2. $\mathcal{A} \setminus \mathcal{B} \in \Omega$ for each finite family of c.e. sets $\mathcal{B}$.

A c.e. set W is maximal if for any coinfinite c.e. set B with $W \subseteq B$ the difference $B \setminus W$ is finite.

Atoms and Coatoms of $\Omega^X_{/I^X_F}$

W is maximal $\Rightarrow W^*$ is a coatom of $\mathscr{E}^*$.

Theorem

For any infinite family $\mathscr{A} \in \Omega^X$ there exists a family $\mathscr{B} \subseteq \mathscr{A} \cup \{\mathbb{N}\}$ such that $\mathscr{B} \in \Omega^X$ and $\mathscr{A} \setminus \mathscr{B}$ is infinite.

Theorem

Let $\mathscr{S} \in \Omega^X$ be a coinfinite family containing all finite sets such that $\{e : W^A_e \in \mathscr{S}\} \leqslant_T X''$. Then there exists a coinfinite family $\mathscr{A} \in \Omega^X$ such that $\mathscr{S} \subseteq \mathscr{A}$ and for any coinfinite family $\mathscr{B}$ with $\mathscr{A} \subseteq \mathscr{B}$ the difference $\mathscr{B} \setminus \mathscr{A}$ is finite.

$\Omega^X_{/I^X_F}$ does not contain atoms, but contains coatoms.

Subtrahends

A family $\mathscr{A} \in \Omega^X$ is a subtrahend if $\mathscr{B} \setminus \mathscr{A} \in \Omega^X$ for any $\mathscr{B} \in \Omega^X$.
Let Ω_S^X be the class of all subtrahends.

Ω_S^X is also definable in Ω^X.

Theorem (Degtev, 1991)

Let $\mathscr{A} \in \mathrm{Fin}^X$. Then $\mathscr{A} \in \Omega_S^X$ iff any set $F \in \mathscr{A}$ is finite.

Corollary

1. $D^X = \{\mathscr{A} \in \Omega^X : \forall F \in \mathscr{A}\, [F \text{ is finite}]\}$ is definable in Ω^X.
2. The singleton $\mathscr{F} = \{F \subseteq \mathbb{N} : F \text{ is finite}\}$ is definable in Ω^X.

Indeed, $\mathscr{A} \in D^X$ iff $\forall \mathscr{C} \in \mathrm{Fin}^X\, [\mathscr{C} \subseteq \mathscr{A} \Rightarrow \mathscr{C} \in \Omega_S^X]$,

$\mathscr{B} = \mathscr{F}$ iff $\mathscr{B} \in D^X \;\&\; \forall \mathscr{C} \in D^X\, [\mathscr{C} \subseteq \mathscr{B}]$.

Weak Minuends

A family $\mathscr{A} \in \Omega$ is a weak minuend if $\mathscr{A} \cap \mathscr{B} \in \Omega$ for any $\mathscr{B} \in \Omega$. Let Ω_{WM} be the class of all weak minuends. Note that $\Omega_M = \mathrm{Fin} \subseteq \Omega_{WM}$.

A family $\mathscr{A} \in \Omega$ is called a completely c.e. if the index set of $\mathscr{A}$ is c.e. By Rice-Shapiro Theorem, $\mathscr{A} \neq \emptyset$ is completely c.e. iff $\mathscr{A} = \{X : D_{f(x)} \subseteq X,\ x \in \mathbb{N}\}$ for some computable function f.

Proposition (Degtev, 1991)

If $\mathscr{A}$ is completely c.e., then $\mathscr{A} \in \Omega_{WM}$.

Question

Is there a family $\mathscr{A} \in \Omega_{WM}$ such that $\mathscr{A} \neq^* \mathscr{B}$ for any completely c.e. family $\mathscr{B}$ (where $\mathscr{A} =^* \mathscr{B}$ means that $\mathscr{A} \Delta \mathscr{B}$ is finite)?

Thank you for attention!