# Calibrating word problems of groups via the complexity of equivalence relations

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# Three results connecting eqrels and f.g. groups

(1) There is a finitely presented group with a word problem which is a uniformly effectively inseparable equivalence relation.

(2) There is a finitely generated group of computable permutations with a word problem which is a universal co-computably enumerable equivalence relation.

(3) Each c.e. truth-table degree contains the word problem of a finitely generated group of computable permutations.

Main reference for this talk: eponymous 2018 paper by Nies and Sorbi in Math. Struct. in Comp. Science [8].

# Definition of computable reducibility and universality

**Definition**

Given two equivalence relations $R, S$ on $\mathbb{N}$, we say that $R$ is computably reducible to $S$ (notation: $R \leq S$) if there exists a computable function $f$ such that, for every $x, y \in \mathbb{N}$,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

**Definition**

Let $\mathcal{A}$ be a class of equivalence relations. An equivalence relation $R \in \mathcal{A}$ is called $\mathcal{A}$-universal if $S \leq R$ for every $S \in \mathcal{A}$.

# Sample results on universal eqrels

- The isomorphism relation for various familiar classes of computable structures is $\Sigma^1_1$-universal: e.g. computable graphs (Fokina et al. 2012 [4]).
- $1$-equivalence among c.e. sets is $\Sigma^0_3$-universal. (Fokina, Friedman and Nies 2012 [3]).
- Equality of functions $\Sigma^* \to \Sigma^*$ that are computable in quadratic time is a $\Pi^0_1$-universal equivalence relation. The functions are described by Turing programs. Ianovski et al. 2014 [5, Theorem 3.5].
- In contrast, Ianovski et al. show that there is no $\Pi^0_n$-universal equivalence relation for $n > 1$.
- In fact, for $n > 1$, each $\Pi^0_n$ equivalence relation $R$ there is a $\Delta^0_n$ relation $S$ such that $S \not\leq R$.

In the talk, we will discuss four results that relate

$\Sigma^0_1$ universality and $\Pi^0_1$ universality
for equivalence relations

to

word problems and isomorphism problems
for finitely generated groups.

# $\Sigma_1^0$-universal equivalence relations and isomorphism of finitely presented groups

# A little-known construction by C.F. Miller III

Write $F_X$ for the free group on generators in $X$.

**Some notation**

- Given a group $G = \langle X; R \rangle = F_X/N$ where $N$ is the normal closure of the set of relators $R$, the word problem is $\{(s,t) \colon st^{-1} \in N\}$.
- Write $=_G$ for this equivalence relation on $F_X$.

**Theorem (C.F. Miller III, Group theoretic dec. problems, 1971[6])**

(a) Given a $\Sigma^0_1$ eqrel $E$, one can effectively build a f.p. group $G_E = \langle X; R \rangle$, and a computable sequence of words $(w_i)_{i \in \mathbb{N}}$ in $F_X$ such that $i \; E \; k \Leftrightarrow w_i =_G w_k$.

(b) Given a finite presentation $\langle X; R \rangle$ of a group $G$ one can effectively find a computable family $(H^G_w)_{w \in F_X}$ of f.p. groups such that $v =_G w \Leftrightarrow H^G_v \cong H^G_w$ for all $v, w \in F_X$.

# Finitely presented groups and $\Sigma^0_1$-universality

**Corollary (to Miller's Theorem)**

(i) There exists a f.p. group $G$ such that $=_G$ is a $\Sigma^0_1$-universal eqrel.

(ii) The isomorphism relation $\cong_{f.p.}$ between finite presentations of groups is a $\Sigma^0_1$-universal eqrel.

Ianovski, Miller, Ng. and N. 2014 had asked (ii), not knowing that it had already been answered in the affirmative in [6].

Proof. Let $E$ be a $\Sigma^0_1$-universal eqrel. Then

(i) by (a) of Miller's theorem, $E$ is computably reducible to $=_{G_E}$, and thus $=_{G_E}$ is $\Sigma^0_1$-universal;

(ii) by (b) of Miller's theorem, $i \; E \; k \Leftrightarrow H^{G_E}_{w_i} \cong H^{G_E}_{w_k}$. This shows that $E$ is computably reducible to $\cong_{f.p.}$. Hence $\cong_{f.p.}$ is $\Sigma^0_1$-universal.

# A further question on $\cong_{\text{f.p.}}$ answered

- N. and Sorbi 2018 asked whether each pair of distinct equivalence classes of $\cong_{\text{f.p.}}$ is recursively inseparable.
- A negative answer was observed by Maurice Chiodo.

$G_{ab} = G/G'$ is the largest abelian quotient of a group $G$.

**Observation by M. Chiodo, See Lyndon/Schupp, Logic Blog 2017, p. 18.**

Let $\mathcal{A}$ be the set of finite presentations of groups $G$ such that $G_{ab} \cong \mathbb{Z}$. This set $\mathcal{A}$ is recursive.

$\mathcal{A}$ contains all the presentations of $\mathbb{Z}$ and no presentation of $\mathbb{Z} \times \mathbb{Z}$. So these two equivalence classes can be separated by a recursive set.

# A better version of the N. and Sorbi 2018 question

A group $G$ is called perfect if $G' = G$. The finite presentations of perfect groups can be listed effectively. So there is a computable function $P$ such that $P(n) = \langle X_n, R_n \rangle$ is a list of the finite presentations of perfect groups.

Let $E_P = \{\langle n, k \rangle \colon P(n) \cong P(k)\}$. Are any two equivalence classes of $E_P$ recursively inseparable? If so, is $E_P$ uniformly e.i.?

If $Q$ is another such listing then $E_P$ and $E_Q$ are recursively isomorphic. So the answers don't depend on the choice of $P$. (Use a back and forth argument, together with the fact that $\cong_{\text{f.p.}}$ is $\Sigma^0_1$.)
If $E_P$ is u.e.i. then it is already recursively isomorphic to $\sim_{PA}$.
This is because $E_P$ has a "strong diagonal function", i.e.
a computable function $g$ taking finite sets $D \subseteq \mathbb{N}$ as arguments such that $g(D) \notin [D]_E$ for each $D$.

# $\Sigma^0_1$-universal equivalence relations and word problems

# A f.p. group with u.e.i. word problem

**Definition (See Soare 89, Exercise II.4.5)**

Disjoint c.e. sets $A, B$ are effectively inseparable if for each disjoint pair $X, Y$ of c.e. sets, there is a 1-1 computable function $f$ such that $f(X) \subseteq A$ and $f(Y) \subseteq B$. It suffices to ask this for the pair $X = \{e \colon \phi_e(e) = 0\}$, $Y = \{e \colon \phi_e(e) = 1\}$.

**Theorem (First result in N. and Sorbi, 2018)**

There is a finitely presented group $H$ such that each pair of distinct equivalence classes of its word problem $=_H$ is effectively inseparable, uniformly in terms of elements of $F_n$ representing the equivalence classes.

Note that the word problem of $H$ is a $\Sigma^0_1$-universal eqrel by Andrews et al. 2014 [1].

**Theorem (Recall, f.p. group with u.e.i. WP)**

There is a finitely presented group $H$ such that each pair of distinct equivalence classes of its word problem $=_D$ is effectively inseparable in a uniform way.

The proof has three main ingredients. (See the paper for detail.)

1. Lemma. Let $G = \langle X; R \rangle$ be a given f.p. group. Suppose $([1]_G, [w]_G)$ is e.i. where $w \in F_X$. Let $N = \mathtt{Ncl}_G(w)$. Then, if $s, t \in N$ such that $s \neq_G t$, the pair $([s]_G, [t]_G)$ is e.i. uniformly in $s, t$.

2. A method of C.F. Miller builds a nontrivial f.p. group so that all its nontrivial quotients have an undecidable WP. This is done by encoding an e.i. pair into the word problem.

3. A construction from Lyndon/Schupp IV.3.5. embeds each countable group into a f.g. simple group.

$\Pi^0_1$-universal equivalence relations

and word problems

# F.g. subgroups of $S_{\mathrm{rec}}$

Let $\alpha, \beta$ of two permutations on some set $W$. Then $\alpha\beta$ denotes the permutation such that $\alpha\beta(s) = \beta(\alpha(s))$ where $s \in W$.
Let $S_{\mathrm{rec}}$ denote the group of computable permutations of $\mathbb{N}$.

**Fact**

Suppose $G$ is a f.g. subgroup of $S_{\mathrm{rec}}$. Then the WP of $G$ is $\Pi^0_1$.

**Fact**

Suppose that a f.g. group $G$ has decidable WP. Then $G$ is isomorphic to a subgroup of $S_{\mathrm{rec}}$. (Use the right translation action of the generators.)

In contrast, Morozov 2000 [7] showed that there is a two-generator group with $\Pi^0_1$ word problem that is not embeddable into the group of computable permutations of $\mathbb{N}$.

# Downwd closed classes of finitely generated groups

In the diagram below, arrows denote proper inclusions. All its classes of f.g. groups are closed under taking subgroups.

$$\begin{tikzcd}
\Sigma^0_1 \text{ word problem} & & \Pi^0_1 \text{ word problem} \\
& & \text{subgroup of } S_{\mathrm{rec}} \arrow[u] \\
& \text{Computable WP} \arrow[uul] \arrow[ur] &
\end{tikzcd}$$

# Automorphisms of negative numerations

Let $\nu\colon \mathbb{N} \to M$ be a numeration. Call a permutation $\rho$ on $M$ computable if there are computable $f, g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\rho \circ \nu = \nu \circ f \text{ and } \rho^{-1} \circ \nu = \nu \circ g.$$

I.e., $f$ "names" $\rho$ and $g$ "names" $\rho^{-1}$ w.r.t. $\nu$. These permutations form a group denoted $S_{\mathrm{rec}}(\nu)$.

FACT. If $\nu$ is a negative numeration (i.e. its kernel is $\Pi^0_1$) then each f.g. subgroup $G$ of $S_{\mathrm{rec}}(\nu)$ has $\Pi^0_1$ word problem.

FACT. There is a single negative numeration $\nu$ such that each f.g. group with $\Pi^0_1$ WP occurs as a subgroup of $S_{\mathrm{rec}}(\nu)$.

To verify the second fact, one combines Morozov 2000 [7] (where the negative numeration depends on $G$) with the construction of a $\Pi^0_1$ universal eqrel in Ianovski et al. 2014 [5].

**Theorem (N. and Sorbi 2018, second result)**

There is a finitely generated group of computable permutations of $\mathbb{N}$ with word problem a $\Pi^0_1$-universal equivalence relation.

To prove this, let $E$ be a $\Pi^0_1$-universal equivalence relation (Ianovski et al. [5]). By [5, Prop. 3.1] there is a computable function $f$ such that

$$x \; E \; y \Leftrightarrow (\forall n)[f(x,n) = f(y,n)].$$

The construction of $f$ shows that $f(x,n) \leq x$ for each $x, n$.

# Basic setting for the proof

Fix a computable bijection $\langle \cdot\,,\,\cdot \rangle \colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. The domain of our computable permutations is a disjoint union of pairs of "columns"

$$C_x^i = \{2x + i\} \times \mathbb{N},$$

where $i = 0, 1$ and $x \in \mathbb{Z}$ for the rest of this proof.
The permutation $\sigma$ shifts $C_x^i$ to $C_{x+1}^i$:

$$\sigma(\langle 2x + i, n \rangle) = \langle 2x + 2 + i, n \rangle.$$

The permutation $\tau$ exchanges $C_0^i$ with $C_0^{1-i}$ and is the identity elsewhere:

$$\tau(\langle i, n \rangle) = \langle 1 - i, n \rangle \text{ and } \tau(\langle k, n \rangle) = \langle k, n \rangle \text{ if } k \neq 0, 1.$$

# The permutation $\alpha$ encoding $f$

Recall that $E$ is $\Pi^0_1$ universal eqrel, and $f$ is computable, s.t.

$$x \mathrel{E} y \Leftrightarrow (\forall n)[f(x,n) = f(y,n)].$$

The permutation $\alpha$ codes $f$ in the sense that there exists a fixed computable sequence $(t_x)_{x\in\mathbb{N}}$ of terms in the free group generated by the symbols $\alpha, \sigma, \tau$, such that,
letting $G = \langle \alpha, \sigma, \tau \rangle \leq S_{\mathrm{rec}}$, for each $x, y \in \mathbb{N}$ we have

$$\forall n\,[f(x,n) = f(y,n)] \Leftrightarrow t_x =_G t_y. \tag{1}$$

For each $x, n$,

- $\alpha$ has a cycle of length $f(x,n)+1$ in the interval $[n(x+1), n(x+1)+x]$ of $C^0_x$
- $\alpha$ is the identity on the remaining columns.

# Defining terms $t_x(\alpha, \sigma, \tau)$

For $x \in \mathbb{N}$ we let $t_x = \sigma^x \alpha \sigma^{-x} \ \tau \ \sigma^x \alpha^{-1} \sigma^{-x}$.

- the permutation $t_x(\alpha, \sigma, \tau)$ only retains the encoding of the values $f(x, n)$, and erases all other information:
- it moves this information to the pair of columns $C_0^0, C_0^1$. In this way we can compare the values $f(x, n)$ and $f(y, n)$ applying $t_x$ and $t_y$ to $\alpha, \sigma, \tau$:

$$\forall n \, [f(x, n) = f(y, n)] \ \Leftrightarrow \ t_x = t_y.$$

In more detail, et $\alpha_x$ be the permutation given by $\alpha(\langle 2x, w \rangle) = \langle 2x, \alpha_x(w) \rangle$. We obtain

$$t_x(\langle u, w \rangle) = \begin{cases} \langle u, w \rangle, & \text{if } u \neq 0, 1, \\ \langle 1, \alpha_x(w) \rangle, & \text{if } u = 0, \\ \langle 0, (\alpha_x)^{-1}(w) \rangle, & \text{if } u = 1. \end{cases}$$

# Background for the final result in N. and Sorbi 2018

For the rest of the talk, the "word problem" of a group $G = F_n/N$ is meant classically as the equivalence class of the identity element, i.e. $N$.

- Collins 1971 [2] showed that each r.e. truth table degree contains the word problem of a finitely presented group, extending the work of Fridman, Clapham, Boone and others showing this for c.e. Turing degrees.
- In contrast, Ziegler 1976 [9] constructed an r.e. bounded truth-table degree that does not contain the word problem of a finitely presented group.

# Analog of Collins' result for $\Pi^0_1$ groups

Let us call a permutation $\sigma$ fully primitive recursive if

both $\sigma$ and $\sigma^{-1}$ are primitive recursive.

The fully primitive recursive permutations form a group.

**Theorem**

Given an r.e. set $S$, there is a triple of fully primitive recursive permutations such that the group $G$ generated by them has word problem truth table equivalent to $S$.

We prove this by modifying the construction of computable permutations $\alpha, \sigma, \tau$ for our previous result.

# Open questions

1. Is isomorphism of f.p. perfect groups an u.e.i. equivalence relation?

2. It there a f.g. group with u.e.i. word problem that also has a strong diagonal function? I.e., can the WP be recursively isomorphic to $\sim_{PA}$?

The third question connects $\Pi^0_1$ universality with a different area. It was asked by Ianovski, Miller, Ng and N. 2014 [5] and remains open to my knowledge.

3. Is isomorphism of finite-automata presentable equivalence relations $\Pi^0_1$-universal?

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